BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW OF MATHEMATICAL SCIENCE

EDITED BY

F. N. COLE VIRGIL SNYDER J. W. YOUNG
ALEXANDER ZIWET D. E. SMITH
T. LEVI-CIVITA R. C. ARCHIBALD

VOL. XXVI

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BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

IN MEMORY OF GABRIEL MARCUS GREEN.

BY PROFESSOR E. J. WILCZYNSKI.

(Read before the American Mathematical Society March 28, 1919).

GABRIEL MARCUS GREEN was born in the city of New York on October 19, 1891. He died on January 24, 1919, at Cambridge, Massachusetts. In this brief span of years he has won enduring fame. In spite of his extreme youth, we mourn in him not the promise of a genius unfulfilled, but the sad untimely loss of a great leader of proven strength whose power and insight had been fully tested, and whose actual achievements can never perish.

Green graduated from Public School No. 4 of New York in 1904 as valedictorian of his class. He then entered the high-school department of the College of the City of New York, and graduated from the College in 1911 at the head of his class. In 1909 he received the Belden Mathematical Prize, in 1909 and 1910 the Pell Medal for the highest rank in all subjects, and in 1910 and 1911 the Kenyon Prize for distinction in pure and applied mathematics. He then entered Columbia University as a graduate student, received the degree of Master of Arts in 1912, and the doctor's degree in 1913. He was a member of Phi Beta Kappa and of Sigma Xi.

While at Columbia, Green became a member of Edward Kasner's seminar. It was here that his interest in projective differential geometry was first aroused, an interest which never flagged and which dominated his whole mathematical career.

In 1913, Dr. Green acted as instructor in mathematics at the College of the City of New York. In 1914 he was appointed to an instructorship at Harvard University, and in

1916 became a member of the faculty. He was a clear, interesting, and inspiring teacher. The department was unanimous in its wish that Green should be promoted to an assistant professorship, and the matter would shortly have been laid before the corporation. But a sudden attack of influenza, followed by pneumonia, cut short this life, so valuable to science and so precious to his friends, before any action could be taken.

Green's musical gifts were hardly inferior to his mathematical talent. He was entirely self taught and was much hampered, in his earlier years, by the inadequacy of the piano which belonged to his parents. Nevertheless, diligent practice on this instrument and a dumb keyboard enabled him to acquire an extensive repertoire and a remarkable technique. His touch was delicate and his musical intuition fine. Music was a form of expression especially well adapted to his emotional and idealistic temperament.

He was short of stature and slender; his features indicated strength and refinement; his expression, always sensitive and often serious, had in it a characteristic undertone of cheerfulness and joy, the joy of a man whose faith in life had not been destroyed, and whose belief in his own powers had not been broken.

Green's first published paper is his Columbia dissertation entitled "Projective differential geometry of triple systems of surfaces," printed privately at Lancaster, Pa., in 1913. The way he wrote and published this thesis is characteristic of his independence and modesty. It had been my task for several years to keep myself informed as to the status and the progress of projective differential geometry; and I had thought that, in the United States at least, no one was likely to be engaged actively in that field without my knowledge. It was with great surprise, then, that I read of a paper with the above title presented to the American Mathematical Society at New York. The title would have aroused my interest at any time, but I had good reason to take more than ordinary interest in this paper just then. I had been engaged for some time in studying this very subject and was nearly ready to prepare my results for publication. I wrote to Green, asking him for some of the details of his work, not suspecting that it had been published already, and to my great astonishment received, a few days later, his printed thesis. This thesis

made it quite unnecessary for me to publish my own work on the subject.

It is unfortunate that this thesis was printed privately; it ought to be reprinted in one of our journals, so as to render its contents generally available. The importance of the subject may be gauged by noting that it may be regarded as containing, among many other things, the theory of triply orthogonal systems, to which subject alone Darboux has devoted a volume of 567 pages.

The theory of triple systems of surfaces, as developed by Green, is based on the theory of invariants and covariants of a completely integrable system of six linear partial differential equations of the second order with one dependent and three independent variables. The subject is far too vast to be treated exhaustively in a thesis of twenty-seven pages; but the fundamental aspects of the theory are there developed and further work in this important field must inevitably set in where Green left off or else duplicate his work.

Closely connected with this thesis are two papers published in 1915 and 1916 in the *American Journal of Mathematics* on "One-parameter families of space curves and conjugate nets on a curved surface." Of course, a continuous one-parameter family of space curves generates a surface, and the properties of this surface are therefore also properties of the one-parameter family of curves. Moreover, these curves determine a second one-parameter family of curves on the surface, such that the two families together form a conjugate system. Finally each of these families determines a congruence, composed of the tangents of the curves of which it is composed. Consequently the theory of a one-parameter family of space curves, the theory of conjugate systems, the theory of surfaces, and the theory of congruences, although distinct, are all intimately connected. In formulating these connections and these distinctions analytically Green exhibits a complete mastery of the situation. The formulas developed by him in his first memoir, the paper of 1915, apply directly to any one-parameter family of curves which are not asymptotic lines on the surface S determined by them. The reason for this exception is easily understood. He uses two independent variables, u and v, one of which, namely v, is constant along one of the curves of the given one-parameter family, while the other changes along such a curve. In every case

then, there presents itself a new one-parameter family of curves on the surface S, the curves $u = \text{const.}$ To simplify the analysis, Green chooses the curves $u = \text{const.}$ to be the curves on S which are conjugate to the given curves $v = \text{const.}$ This can always be done except when the given curves $v = \text{const.}$ are asymptotic curves of S, since asymptotic curves are their own conjugates. Nothing vital is lost by this procedure since the exceptional case of a one-parameter family of space curves, composed of the asymptotic lines of a surface, may be regarded as completely settled by a previously developed theory.

We have purposely discussed this paper, published in 1915, before mentioning an earlier paper of Green's on "One-parameter families of curves in the plane" (*Transactions*, 1914). There was then in existence a closely related theory whose basal features may be described in a very few words. Let $y^{(1)}, y^{(2)}, y^{(3)}$ be the homogeneous coordinates of a point P_y in the plane. A one-parameter family of plane curves may be represented in the form

$$y^{(k)} = f^{(k)}(u, v), \qquad (k = 1, 2, 3),$$

where u is the variable which changes when P_y moves along a curve of the given one-parameter family, while v remains constant. Thus v represents the parameter of the family and changes when we pass from one curve of the family to another; the curves of the given family are the curves $v = \text{const.}$ But when formulated in this way, the fact that these functions $f^{(k)}(u, v)$ depend upon both u and v leads one to consider the curves $u = \text{const.}$ as well. The result is a study, not so much of the given one-parameter family itself, as of two one-parameter families, and the net composed of both. This was essentially the situation in which Green found the theory of one-parameter families of plane curves.

In general the second one-parameter family introduced in this way is entirely independent of the first family, and one might say that the properties of the first family are those properties of the net which remain unaltered if the second one-parameter family is changed in all possible ways. A second method of obtaining a theory pertaining entirely to the given one-parameter family of curves would be to show that there exists a second one-parameter family of plane curves completely determined by the first, and then to study

the net composed of the two families. This is precisely what Green did in his theory of one-parameter families of *space* curves by associating, with the given family, the conjugate system. But strange to say, in the theory of *plane* curves, this was a far more difficult matter. The notion of conjugacy breaks down in that case; the idea which might suggest itself of associating with the given curves their orthogonal or certain other isogonal trajectories to constitute a net has to be discarded, since isogonality is a notion not admissible in projective geometry. But Green was able to conquer these difficulties by showing that every one-parameter family of plane curves determines uniquely a second one-parameter family by means of a certain purely projective relation; the theory of the canonical net, as he calls it, which is determined in this way, is then equivalent to the theory of the one-parameter family of plane curves.

In both of the papers just mentioned, Green discusses a question which is of considerable practical importance. It often happens that a theory of invariants has been developed which may be regarded as being general, in so far as it applies to every form of a certain class, but whose practical applicability nevertheless seems doubtful because the formal theory was based, not on the explicitly most general form of its class, but upon a certain canonical form to which every form of the class may be reduced. The obvious way to compute the invariants of such a form would be to reduce it to its canonical form and then to use the formulas for the invariants which the existing theory supplies. But this method cannot always be carried out because the reduction to the canonical form, while possible theoretically, may require operations, like integrations of differential equations, which cannot be carried out in practice.

Such, for instance, was the situation in the projective theory of surfaces at the time when Green wrote these papers. The general formulas were based on the assumption that the surface was referred to its asymptotic lines and, although a method had been given for computing the invariants of a surface when the asymptotic lines were not known explicitly, this method was difficult to apply to special cases. The possibility of using a different method was apparent. Although the actual determination of the asymptotic lines of a surface requires the integration of a quadratic differential

equation of the first order, it is evident that the invariants of the surface must be capable of expression in terms of quantities which define the surface itself rather than its asymptotic lines; they must therefore have expressions which can be made explicit without actually integrating the differential equation of the asymptotic lines. It is Green's merit not only to have appreciated this point, which could not escape him or others, but to have shown that these expressions can be written out in a comparatively simple form. He has actually obtained these general formulas in his paper "On the theory of curved surfaces, and canonical systems in projective differential geometry," *Transactions*, 1915.

The second memoir on "Projective differential geometry of one-parameter families of space curves and conjugate nets on a curved surface" contains a number of interesting and important results. I shall speak in detail merely about one of them. It is well known that the classical metric theory of surfaces may be regarded as the invariant theory of two simultaneous quadratic differential forms which are usually written in the form

$$Edu^2 + 2Fdudv + Gdv^2$$

and

$$Ddu^2 + 2D'dudv + D''dv^2.$$

If $F = 0$, and if the ratio $E : G$ can be expressed as the product of a function of u alone by a function of v alone, the curves $u = \text{const.}$ and $v = \text{const.}$ are said to form an orthogonal isothermal system. If $D' = 0$, the curves $u = \text{const.}$ and $v = \text{const.}$ form a conjugate system. If, besides, the ratio $D : D''$ can be expressed as the product of a function of u alone by a function of v alone, the curves $u = \text{const.}$ and $v = \text{const.}$ are said to be isothermally conjugate, a terminology introduced by Bianchi and justified by the fact that these systems are related to the second differential form in the same way as isothermal orthogonal systems are to the first. Now Bianchi had shown, many years ago, that if a system of curves forms an isothermally conjugate system, this property will be preserved under all projective transformations. It is therefore a projective property. But what is this property? The definition quoted above was purely analytic, and no one had the slightest idea as to what it really meant. Nevertheless this property, whose geometric significance was entirely

hidden, was entering into a continually growing chain of theorems. Could it be admitted under the circumstances that the meaning of any of these theorems was really known? Here then there was a most embarrassing situation. And since the property was known to be of a projective character, the duty of obtaining the desired interpretation clearly devolved upon projective differential geometry. In 1915 I succeeded in finding a simple algebraic relation between certain completely interpreted invariants which was satisfied if and only if the system was isothermally conjugate. But I never felt completely satisfied with my solution of the problem. A great step toward the real solution was taken by Green.

In order to make intelligible Green's condition for isothermal conjugacy, a few preliminary explanations are necessary. Let us remember in the first place that we are dealing with a conjugate system of curves on a surface S. Let P be a point of this surface; let t and t' be the tangents at P of the two curves of the conjugate systems which pass through P; let a and a' be the asymptotic tangents of the point P. Then t and t' are separated harmonically by a and a'; this follows immediately from the definition of a conjugate system. Consider the osculating plane at P of each of the two curves of the conjugate system passing through P. They intersect in a line l called the axis of the point P. Thus there is associated with every point P of S a line through P, namely, its axis. The system of ∞^2 lines associated in this way with all of the points of a surface is called the axis congruence. The lines of a congruence can be assembled into a one-parameter family of developables in two ways. Consequently there will pass through P two curves, called axis curves, such that upon each of them the axes of all of its points form developables.

The given conjugate system of curves is composed of two one-parameter families. The tangents of each of these families form a congruence of lines which has S for one sheet of its focal surface. Let P_1 be the point where the tangent t touches the second sheet of the focal surface of the congruence to which it belongs. Let us define P_{-1} similarly on t': The line P_1P_{-1} is called the ray of the point P. The system of ∞^2 rays which correspond to the ∞^2 points of S is called the ray congruence. The curves on S which correspond to the developables of the ray congruence are its ray curves. Thus we have at a general point of our surface, the two asymptotic

tangents a and a', the tangents of the given conjugate system t and t', the pair of axis tangents, and the pair of ray tangents. All of these notions had been formulated by Green's predecessors, although he had been led to consider the axis congruence independently. Now Green defines his *anti-ray tangents*. They are the harmonic conjugates of the ray-curve tangents with respect to t and t'. Moreover there exists a uniquely determined pair of lines which separate both of the pairs a and a', and t and t' harmonically. These had been considered before, but Green first pointed out their great importance. He calls them the *associate conjugate* tangents:

We have now explained all of the terms in Green's theorem. *A necessary and sufficient condition that a conjugate net on a surface be isothermally conjugate is that for each point of the surface the pair of axis tangents, the pair of anti-ray tangents, and the pair of associate conjugate tangents form pairs of the same involution.* It should be noted however that there is a case, which seems to have escaped Green's attention, which is not settled by this theorem. The exceptional case arises when the axis tangents, and therefore the ray-tangents as well as the anti-ray tangents, separate the tangents of the fundamental conjugate system harmonically. It remains to find a substitute for Green's theorem to cover this case.

I have gone into such details on this point because this theorem, it seems to me, shows better than any other single one of his theorems, the penetrating power of Green's mind, and his fine feeling for mathematical elegance. Perhaps a still simpler geometric condition for isothermal conjugacy will be found; but I am inclined to believe that Green's theorem, in so far as it is applicable, is the final form for this condition.

I now proceed to another paper, also published in 1916, which shows very clearly his great power of generalization. This is his paper on "The linear dependence of functions of several variables and completely integrable systems of homogeneous linear partial differential equations."

It is a familiar fact that $n+1$ analytic functions $y_1, y_2, \cdots, y_n, y_{n+1}$ of a single variable are linearly dependent, if and only if their Wronskian is identically equal to zero. If, at the same time, the Wronskian of $y_1, y_2, \cdots, y_n$ is not zero, $y_1, y_2, \cdots, y_n$ will be fundamental solutions of a linear homogeneous differential equation

$$(D) \qquad \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0,$$

and the Wronskian of $y_1, \cdots, y_n$ is capable of a simple expression in terms of the coefficients of this differential equation; in fact it can be expressed in terms of p_1 alone, namely, as a simple exponential function of $\int p_1 dx$, a fact first noted by Abel. A transformation of the form $y = \lambda(x)\bar{y}$ can always be made which will reduce the differential equation (D) to another one of the same form but with a vanishing coefficient for $d^{n-1}\bar{y}/dx^{n-1}$, the so-called canonical form of the differential equation, and this canonical form is unique. It then appears that the remaining coefficients of the canonical form are invariants of (D) under all transformations of the form $y = \lambda(x)\bar{y}$, where λ is an arbitrary function of x.

The analytic discussions of various problems of projective differential geometry had shown clearly in every one of the cases which had actually been investigated that a similar situation exists whenever we consider systems of linearly independent functions of several independent variables, and associate them with completely integrable systems of linear homogeneous partial differential equations. But it required deep insight and a very considerable power of generalization to reveal that the situation thus indicated by the various special theories was actually a universal one. This was accomplished by Green in a most satisfactory way without moreover restricting himself to the case of analytic functions.* It was previously known that the projective differential geometry of a k-spread in a space of any number of dimensions is equivalent to the theory of the invariants and covariants of a certain completely integrable system of linear partial differential equations. Green's theorem, however, enables us to state further that the first stage in the calculation of these invariants, namely the determination of the so-called seminvariants, may always be accomplished explicitly by the same methods which have succeeded in the special cases which have already been studied.

The latest results of Green's geometric studies are closely connected with a relation which he has called the relation R. Green's attention seems to have been called to this relation by the relation between the axis and ray congruences and by a

* In this connection paper No. 9 of the appended list is of special interest.

remarkable fact which presents itself in the theory of surfaces and which I shall recall in a few words. Let us consider a point P on a non-developable surface S and the two asymptotic curves a_1 and a_2 which cross at P. The osculating linear complexes, C_1 and C_2, of these curves have a linear congruence in common. The directrices, d_1 and d_2, of this congruence have the property that one of them, d_1, lies in the plane tangent to S at P but does not pass through P, while the other, d_2, passes through P but does not lie in the tangent plane. These two lines, d_1 and d_2, are called the directrices of the first and second kind respectively of the point P. As P moves over the surface S, they generate the corresponding directrix congruences. The curves, called directrix curves, on S which correspond to the developables of these two congruences, are the same for both; i.e., the developables of the two directrix congruences correspond to each other.

The two directrices, d_1 and d_2, of a surface point are in the relation R, but they furnish only a particular case of this relation. The general relation suggested to Green by this particular case may be explained as follows: Given a surface S and a net composed of two one-parameter families of curves on S, the curves $u = \text{const.}$ and $v = \text{const.}$; let t_1 and t_2 be the tangents of the two curves of the net which meet at P. Given also a line l through every point P of S and not in the corresponding tangent plane. Consider the ruled surface formed by the tangents of the curves $u = \text{const.}$ along a fixed curve $v = \text{const.}$ The plane of l and t_1 will be tangent to this ruled surface at a certain point M_1 of t_1. In similar fashion we determine a point M_2 on t_2. The line M_1M_2, which we shall call l', is in the tangent plane; it is uniquely determined when the net on S and the line l through P are given. Moreover, unless the given net is conjugate, l is also uniquely determined by l'. Two lines related to one another in this way with respect to a net of curves are said by Green to be in the relation R; this nomenclature is also extended to the resulting line congruences. In the particular case when the given net is composed of the asymptotic curves of a non-developable surface, the two lines are polar reciprocals with respect to the osculating quadric of the surface point P. I shall hereafter speak of two lines l and l', which are in the relation R, as Green reciprocals of each other.

I shall mention only a few of the results which Green has

obtained by using this notion. He found an elegant property of isothermal systems which may be formulated as follows: *An orthogonal net on a non-developable surface is an isothermal net, if and only if the developables of the Green reciprocal of the congruence of normals of the surface correspond to a conjugate net on the surface.* He also found the following result which emphasizes the exceptional properties of the directrix congruences. *The two directrix congruences of a surface form the only pair of congruences, whose members are Green reciprocals of each other with respect to the net of asymptotic lines of the surface, and such that the developables of the two congruences correspond to each other.*

The directrix congruences thus have undoubted claims to special attention, and a large number of questions arise when we consider the directrix congruence of the second kind as a kind of projective analogon of the congruence of normals which is so important in the metric theory of surfaces. At one point however, the analogy breaks down. The developables of the normal congruence determine on the surface a net of curves which is not only orthogonal but also conjugate, namely, the lines of curvature. But the directrix curves do not, in general, form a conjugate system. Green has succeeded in defining in purely projective fashion a new pair of congruences, Green reciprocals of each other, one of which may be regarded as being in this sense completely analogous to the normal congruence. He calls it the pseudo-normal congruence and has established its importance in most convincing fashion by a number of applications. One of these is concerned with a fundamental question which has long remained unsolved. Darboux showed, in 1880, that in the neighborhood of a regular point of a surface, the equation of the surface can be expanded in either of the two forms

$$z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}(Ix^4 + Jy^4) + \cdots,$$

$$z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}\,xy(Kx^2 + Ly^2) + \cdots,$$

where x, y, and z denote non-homogeneous projective coordinates with respect to an appropriately chosen tetrahedron, and where I, J, K, and L, are absolute projective invariants of the surface. But Darboux did not explain the relation of the tetrahedron to the surface for either of these expansions. This question was settled by me about ten years ago, for the

first of these two expansions. Green has now found the corresponding interpretation for the second expansion, and in this interpretation the pseudo-normal plays an important part.

Most of the results of which I have just spoken were announced by Green without proof in the form of brief notes published in the *Proceedings* of the National Academy. The proofs are given in a lengthy memoir, which has appeared in the *Transactions*, April, 1919.

My account of Green's work is far from complete. I have attempted to give only a bare outline of some of his most important work. In the six short years of his mathematical career, from 1913–1919, he enriched geometry with so many new ideas and important results as would suffice to excite our admiration if they had been spread over all of a normal life time. In his death we have suffered a heavy loss, but his life and work will continue to be, for many of us, an everlasting source of strength and inspiration.

CHICAGO,
March 29, 1919.

LIST OF GREEN'S PUBLISHED PAPERS.

1913.

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REDUCTION OF THE ELLIPTIC ELEMENT TO THE WEIERSTRASS FORM.

BY PROFESSOR F. H. SAFFORD.

(Read before the American Mathematical Society April 26, 1919.)

In Enneper's Elliptische Functionen, page 27, may be found a method due to Weierstrass of reducing the general elliptic element to the Weierstrass form. Briefly, from

$$(1)\quad x = x_0 + \frac{\sqrt{R(x_0)}\,\sqrt{(4s^3 - g_2 s - g_3)} + \tfrac{1}{2}R'(x_0)[s - \tfrac{1}{24}R''(x_0)] + \tfrac{1}{24}R(x_0)R'''(x_0)}{2[s - \tfrac{1}{24}R''(x_0)]^2 - \tfrac{1}{2}A \cdot R(x_0)}$$

comes

$$(2) \qquad \frac{dx}{\sqrt{R(x)}} = \frac{-\,ds}{\sqrt{(4s^3 - g_2 s - g_3)}} = du.$$

Enneper adds: "Diese Resultate wurden zuerst veröffentlicht von W. Biermann: 'Problemata quædam mechanica functionum ellipticarum ope soluta,' Diss. Berlin, 1864," and the latter states that they were given in Weierstrass's lectures.

Necessary additional formulas are

$$R(x) = Ax^4 + 4Bx^3 + 6Cx^2 + 4B'x + A'$$
$$= r_0 + r_1(x - x_0) + r_2(x - x_0)^2 + r_3(x - x_0)^3 + r_4(x - x_0)^4,$$
$$r_0 = R(x_0), \quad r_1 = R'(x_0), \quad r_2 = R''(x_0)/2, \quad r_3 = R'''(x_0)/6, \quad r_4 = R''''(x_0)/24,$$

$$(3) \qquad g_2 = AA' + 3C^2 - 4BB' = r_0 r_4 - \tfrac{1}{4} r_1 r_3 + \tfrac{1}{12} r_2^2 = 4\epsilon_1\epsilon_2\epsilon_3,$$
$$g_3 = ACA' + 2BCB' - AB'^2 - A'B^2 - C^3$$
$$= \tfrac{1}{6} r_0 r_2 r_4 + \tfrac{1}{48} r_1 r_2 r_3 - \tfrac{1}{16} r_0 r_3^2 - \tfrac{1}{16} r_4 r_1^2 - \tfrac{2}{3}\tfrac{1}{16} r_2^3$$
$$= -\,4(\epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1),$$
$$s = \wp(u + c), \qquad (x_0 \text{ and } c \text{ are arbitrary constants}).$$

Enneper later uses (1) in obtaining the addition formula for $\wp(u + v)$, and Greenhill also quotes (1), adding a brief note.

Haentzschel, in Reduction der Potentialgleichung, discussing surfaces of revolution in a potential problem, obtains, following Wangerin, this differential equation

$$(4) \quad F'^2(u) = AF^4(u) + 4BF^3(u) + 6CF^2(u) + 4B'F(u) + A' = R(F),$$

for which he gives a solution

$$(5) \qquad x = F(u) = \alpha + \frac{R'(\alpha)/4}{s - R''(\alpha)/24},$$

which is obtainable from (1) by using for x_0 the value α, any root of $R(x) = 0$. Haentzschel adds "... so könnte nach Herrn Weierstrass der Ausdruck (5) doch noch nicht als die allgemeinste Form des Integrals von (4) gelten, sondern als solchen leitet Herr Weierstrass ab," and then gives (1) above. Haentzschel uses the simpler solution in obtaining various families of orthogonal curves according to the methods of conjugate functions, but merely gives an outline for the case of the (apparently) more general solution.

The writer has given discussions and extensions of Haentzschel's results in the *Archiv der Mathematik und Physik*. In addition to obtaining the equations of the curves corresponding to the more general solution, he has shown that these equations are factorable and include the curves from the simpler case. These results, omitting computations in general, have appeared in this BULLETIN, June, 1899, March, 1912, November, 1917.

The object of the present paper is to show that solution (1) arises solely from a particular choice of constants in the simpler form (5). But (5) is itself of considerable interest because it may be used to obtain the equation referred to by Durège as "Die Landensche Substitution." If $R(x)$ be replaced by Legendre's form, (5) gives

$$x = \operatorname{sn}(u\sqrt{e}) = \sqrt{(e)(s-\epsilon_1)/(s-\epsilon_2)(s-\epsilon_3)}, \tag{6}$$

while the commonly quoted relation between sn and $\wp$ is

$$\operatorname{sn}(u\sqrt{e}) = \sqrt{e}/\sqrt{(t-a_3)}, \tag{7}$$

in which t and a_3 correspond respectively to s and ϵ_3; in both formulas e is essentially A' used above. From (6) and (7) the substitution may be derived. Proceeding to the more important derivations, the fundamental equation (1) is solved for s, giving

$$s = \wp(u+c)$$

$$= \frac{\sqrt{r_0}\sqrt{R(x)} + r_0 + r_1(x-x_0)/2 + r_2(x-x_0)^2/6}{2(x-x_0)^2}. \tag{8}$$

Next let x_0 have the value α, where α is a root of $R(x) = 0$, so that the corresponding form of (8) is

$$(9)\qquad \begin{aligned} s_1 &= \wp(u + c_1) \\ &= \frac{t_0 + t_1(x-\alpha)/2 + t_2(x-\alpha)^2/6}{2(x-\alpha)^2} \\ &= \frac{t_1/4 + t_2(x-\alpha)/12}{x-\alpha}. \end{aligned}$$

It should be noted that in (9)

$$(10)\qquad t_0 = R(\alpha) = 0, \qquad t_1 = R'(\alpha), \qquad t_2 = R''(\alpha)/2.$$

As c and c_1 are merely constants of integration in solutions of the same differential equation, it is essential to show what relation between them will make the two solutions identical. By the aid of the usual formula

$$(11)\qquad \wp(u \pm v) = \frac{[\wp(u) + \wp(v)][4\wp(u)\wp(v) - g_2] - 2g_3 \mp 2\wp'(u)\wp'(v)}{4(\wp(u) - \wp(v))^2}$$

from (8) and (9) follows

$$(12)\qquad c - c_1 = \wp^{-1}(s) - \wp^{-1}(s_1)$$

or

$$(13)\qquad s_2 = \wp(c - c_1) = \frac{(s + s_1)(4ss_1 - g_2) - 2g_3 - 2s's_1'}{4(s - s_1)^2}.$$

The direct reduction of the second member of (13) would be extremely long, but it can be readily carried far enough by computing the coefficients of certain powers of x to show that

$$(14)\qquad s_2 = \wp(c - c_1) = \frac{r_0 + r_1(\alpha - x_0)/2 + r_2(\alpha - x_0)^2/6}{2(\alpha - x_0)^2},$$

which is the desired relation mentioned above. It is important to notice that s_2 is obtainable from s by replacing x by α, while s_1 was similarly obtained by replacing x_0 by α. In both cases the radicals in s vanish. Though not necessary, a verification of the preceding result, (14), may be effected without difficulty with the aid of (11) and (12) by computing

$$(15)\qquad s = \wp[\wp^{-1}(s_1) + \wp^{-1}(s_2)] = \wp(u + c).$$

Thus (8) or its equivalent form, (1), the fundamental formula due to Weierstrass, is obtainable directly from the simple solution (5) by a suitable choice of the constant c as given by (14).

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A NOTE ON "CONTINUOUS MATHEMATICAL INDUCTION."

BY DR. YUEN REN CHAO.

(Read before the San Francisco Section of the American Mathematical Society April 5, 1919.)

1. *Special case.*—Let the function $f(x)$ be defined in some interval of a real variable x.

Hyp. 1. Let there be a point a in the interval such that $f(a) = 0$.

Hyp. 2. Let there be a constant Δ for the interval, such that $f(x) = 0$ implies $f(x + \delta) = 0$, whenever $0 < \delta \leqq \Delta$.

Then for any b in the interval, where $b > a$, $f(b) = 0$.

Proof.—I. If $b - a \leqq \Delta$, then by Hyp. 2 the conclusion follows.

II. If $b - a > \Delta$, then first apply Archimedes' postulate, that is, there will be an integer n and a fraction $\theta (0 \leqq \theta \leqq 1)$ such that

$$b - a = (n + \theta)\Delta, \qquad \text{or} \qquad b = (a + \theta\Delta) + n\Delta.$$

Next, apply ordinary mathematical induction, thus: By Hyp. 1 and 2, since $\theta\Delta < \Delta$,

$$\therefore \quad f(a + \theta\Delta) = 0.$$

Therefore, by 2, again,

$$f[(a + \theta\Delta) + 1 \cdot \Delta] = 0. \tag{1}$$

By 2, if $f[(a + \theta\Delta) + m \cdot \Delta] = 0$, then

$$f[(a + \theta\Delta) + (m + 1)\Delta] = 0. \tag{2}$$

Hence, combining (1) and (2),

$$f(a + \theta\Delta + n\Delta) = 0,$$

that is,

$$f(b) = 0.$$

2. *General case.*—Let $\varphi(x)$ be any propositional function, defined in some interval of a real variable x.

Hyp. 1. Let there be a point a in the interval such that $\varphi(a)$ is true.

Hyp. 2. Let there be a constant Δ for the interval such that $\varphi(x)$ implies $\varphi(x \pm \delta)$, whenever $0 < \delta \leqq \Delta$.

Then for any b in the interval such that $b \gtreqless a$, respectively, $\varphi(b)$ is true.

The proof will be the same as for the special case, except for obvious changes of wording or sign.

Remarks.—The theorem rests essentially on Archimedes' postulate and on ordinary mathematical induction, but it is not a generalization of the latter, in the sense of including it as a special case. It is not a theorem in mathematical logic, since it is concerned with a real variable x. But it is more general than ordinary theorems dealing with equalities, in that $\varphi(x)$ may be a statement about continuity, convergence, integrability, etc., that cannot be put in the simple form of $f(x) = 0$.

The theorem is a mathematical formulation of the familiar argument from "the thin end of the wedge," or again, the argument from "the camel's nose":

Hyp. 1. Let it be granted that the drinking of half a glass of beer be allowable.

Hyp. 2. If any quantity, x, of beer is allowable, there is no reason why $x + \delta$ is not allowable, so long as δ does not exceed an imperceptible amount Δ.

Therefore any quantity is allowable.

Like all mathematical theorems, the conclusion is no surer than its hypothesis. In this case, if the argument fails, it is usually because a *constant* Δ required in the second hypothesis does not exist. Take the very wedge itself. If a wedge is driven with a constant force between two sides which are pushed together by elastic forces, it will be stopped when balanced by the component of the increasing resistance. In this case the Δ within which δ may increase for $\varphi(x + \delta)$ to continue to hold will not be "uniform for the interval," so to speak, but will become smaller and smaller as x approaches the dangerous point, beyond which the conclusion ceases to be true.

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ON THE NUMBER OF REPRESENTATIONS OF $2n$ AS A SUM OF $2r$ SQUARES.

BY PROFESSOR E. T. BELL.

(Read before the San Francisco Section of the American Mathematical Society April 5, 1919.)

1. OWING possibly to its connection with X-ray analyses of crystal structure, interest in the problem of representing an integer as a sum of integral squares has recently revived. We shall first summarize briefly so much of what is known of the problem as will put the formulas established below in their proper light. Let $N(n, 2r)$ denote the total number of decompositions of n into a sum of $2r$ squares. Then, for $r = 1, 2, 3, 4$, the complete results concerning $N(n, 2r)$ are implicit in sections 40, 41, 42, 65 of the Fundamenta Nova. Jacobi, however, left the explicit statement of all but one of his results to others. When $r \geqslant 4$, $N(n, 2r)$ is expressible in terms of the divisors of n alone. By arithmetical methods, independently of elliptic functions, Eisenstein* proved some of Jacobi's results, showed how the rest might be obtained from his own theorems, and proved that, for $r > 4$ and n general, $N(n, 2r)$ can not be expressed in terms of the divisors of n alone. Letting $\xi_s'(n)$ denote the excess of the sum of the sth powers of all those divisors of n that are of the form $4k + 3$ over the like sum for all divisors of the form $4k + 1$, and $\zeta_s(n)$ the sum of the sth powers of all the divisors, Eisenstein stated a notable exception to his general theorem; showing that at least once, when n is suitably chosen, $N(n, 2r)$, for $r > 4$, may be expressed in terms of $\xi_s'(n)$, or $\zeta_s(n)$. E.g., $N(4k + 3, 10) = 12\xi_4'(4k + 3)$; or what may be shown is ultimately the same thing:† $N(8k + 6, 10) = 204\xi_4'(4k + 3)$. Liouville derived a similar result for $N(2m, 12)$ in terms of $\zeta_5(2m)$. He used for this purpose certain remarkable formulas‡ which, however, he did not prove, and which it is the

* *Crelle*, vol. 35 (1847), p. 135.

† Either result follows from the other on applying a transformation of the second order to the theta equivalent of the appropriate Liouville formula of the kind in § 2.

‡ *Jour. des Math.* (2), vol. 6 (1861); two papers, p. 233, 369 et seq. The formulas for proper representations may be proved similarly to those in this paper.

object of this paper to establish; he also showed how the discovery of such results as Eisenstein's and his own can be made to depend upon the formulas mentioned. In 1907 G. Humbert* and K. Petr† independently proved Eisenstein's 10-square, and Liouville's 12-square results very simply by elliptic functions. At the same time J. W. L. Glaisher‡ published complete results for $N(n, 2r)$, $r = 1$ to 9, inventing the necessary functions for the cases $r = 5$ to 9. He remarked§ that the form of his results seems to indicate the non-existence for $r = 7, 8, 9$ of theorems similar to Eisenstein's or Liouville's for $r = 5, 6$. Recently L. J. Mordell‖ has found and developed close connections between Glaisher's theorems and the elliptic modular invariants. Liouville's formulas seem to have been overlooked by later writers. They offer a direct method of attack upon the question of completely expressing $N(2n, 2r)$ in terms of the divisors of n alone; that is, of finding for what forms of n, r this is possible. The four formulas of Liouville are only the first cases of an infinite number of similar results which may be found as below, using higher powers than the first and second, or products, etc., of the elliptic series; and like results exist for $N(m, 2r)$ where m is odd, Eisenstein's theorem being a consequence of one of these. As Liouville remarks, there is an extensive theory in connection with such formulas. Here we shall merely prove his four, and show how the required coefficients may be found.

2. For m odd, Liouville's formulas are

$$\xi_s(m) \equiv (-1)^{(m+1)/2}\xi_s'(m);$$

$$(1) \qquad \zeta_{2r-1}(m) = \sum_{t=0}^{r-1} A_t N(2m, 4r, 4t+2), \qquad (r > 0);$$

$$(2) \qquad \xi_{2r}(m) = \sum_{t=0}^{r} B_t N(2m, 4r+2, 4t+2), \qquad (r \geqq 0);$$

and for $n = 2^{a+2}m$, m odd, $a \geqq 0$

$$(3) \qquad 2^{(2r+1)a}\zeta_{2r+1}(m) = \sum_{t=0}^{r-1} \alpha_t N(n, 4r+4, 4t+4), \qquad (r > 0);$$

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* *Paris C. R.*, vol. 144, p. 874.

† *Archiv der Math.*, 1907, p. 83.

‡ (1) *Q. J. M.*, vol. 38. The results are summarized in (2): *Proc. L. M. S.* (2), vol. 5 (1907), pp. 479–90.

§ Loc. cit. (2), p. 487, § 13.

‖ *Q. J. M.*, vol. 48 (1917–18), Nos. 189, 190.

$$(4) \qquad 2^{2ar}\xi_{2r}(m) = \sum_{t=0}^{r-1} \beta_t N(n, 4r+2, 4t+4), \qquad (r > 0);$$

where $N(n, r, s)$ = the number of representations of n as a sum of r squares, of which the first s are odd with roots > 0, and the last $r - s$ even with roots $\gtreqless 0$. The coefficients A, B, α, β depend upon r, but not upon m or n. Liouville states that for all values of r

$$A_0 = B_0 = \alpha_0 = \beta_0 = 1;$$

$$C_{r-1} = 16^{r-1}, \qquad C_{r-t-1} = 16^{r-2t-1}C_t,$$

where C denotes either A or α; $B_r > 0$ when $r > 0$; and that recurring formulas exist whereby the successive coefficients may be calculated from the first. These assertions will be verified automatically in the proofs. We shall derive (1) from a comparison of the power series and Fourier developments of $\mathrm{sn}u$; and (2), (3), (4) in a like manner from $\mathrm{cn}u$, sn^2u, $\mathrm{dn}u$ respectively.

3. The necessary series are, with $0! = 1$:

$$(5) \qquad \mathrm{sn}u = \frac{2\pi}{kK}\Sigma\frac{q^{m/2}}{1-q^m}\sin\frac{m\pi u}{2K} = \sum_{r=1}^{\infty} S_{2r-1}(k^2)(-1)^{r-1}\frac{u^{2r-1}}{(2r-1)!};$$

$$(6) \qquad \mathrm{cn}u = \frac{2\pi}{kK}\Sigma\frac{q^{m/2}}{1+q^m}\cos\frac{m\pi u}{2K} = \sum_{r=0}^{\infty} C_{2r}(k^2)(-1)^r\frac{u^{2r}}{(2r)!};$$

$$(7) \qquad \mathrm{dn}u = \frac{2\pi}{K}\left[\frac{1}{4} + \Sigma\frac{q^n}{1+q^{2n}}\cos\frac{n\pi u}{K}\right] = \sum_{r=0}^{\infty} D_{2r}(k^2)(-1)^r\frac{u^{2r}}{(2r)!};$$

$$(8) \qquad \mathrm{sn}^2u = 2\left(\frac{\pi}{kK}\right)^2\left[\Sigma\frac{q^m}{(1-q^m)^2} - \Sigma\frac{nq^n}{1-q^{2n}}\cos\frac{n\pi u}{K}\right] = \sum_{r=1}^{\infty} S_{2r}'(k^2)(-1)^{r-1}\frac{u^{2r}}{(2r)!};$$

the first sums being with respect to $n = 1, 2, 3, 4, \cdots$, and $m = 1, 3, 5, 7, \cdots$. The S, C, D, S' are polynomials in k^2 with positive integral coefficients, and their known forms are

$$S_{2r-1}(k^2) = \sum_{t=0}^{r-1} s_t k^{2t}; \qquad C_{2r}(k^2) = \sum_{t=0}^{r-1} c_t k^{2t};$$

(9)

$$S_{2r}'(k^2) = \sum_{t=0}^{r-1} s_t' k^{2t}; \qquad D_{2r}(k^2) = k^{2r} C_{2r}(1/k^2),$$

the last from the relation $\mathrm{cn}(ku, 1/k) = \mathrm{dn}u$. Also, for all values of r we have *

$$s_0 = c_0 = 1; \qquad s_0' = s'_{r-1} = 2^{2r-1}; \qquad s_{r-t-1} = s_t;$$

(10)

$$s'_{r-t-1} = s_t'; \qquad c_{r-1} = 2^{2r-2}.$$

Henceforth, unless otherwise indicated, n represents an arbitrary integer > 0, m an arbitrary odd integer > 0, and summations are with respect to all m or all n. We shall need the following constants:

$$(11)\quad \Sigma \frac{q^{m/2} m^{2r-1}}{1 - q^m} = \Sigma\, q^{m/2} \zeta_{2r-1}(m); \qquad \Sigma \frac{q^{m/2} m^{2r}}{1 + q^m} = \Sigma\, q^{m/2} \xi_{2r}(m),$$

the right members coming directly from the definitions of ζ, ξ' in § 1, on expanding each term on the left and rearranging all in ascending powers of q. Writing for the moment

$$(12)\quad n = 2^a m, \qquad \xi_r''(n) = 2^{ar} \xi_r(m), \qquad \zeta_r'(n) = 2^{ar} \zeta_r(m),$$

we have also

$$(13)\quad \Sigma \frac{q^n n^{2r}}{1 + q^{2n}} = \Sigma\, q^n \xi_{2r}''(n); \qquad \Sigma \frac{q^n n^{2r+1}}{1 - q^{2n}} = \Sigma\, q^n \zeta'_{2r+1}(n).$$

The necessary theta constants are, $\vartheta_a \equiv \vartheta_a(q)$:

$$(14)\quad \sqrt{\frac{2K}{\pi}} = \vartheta_3 = \sum_{-\infty}^{+\infty} q^{n^2}; \qquad k = \frac{\vartheta_2^2}{\vartheta_3^2}; \qquad \vartheta_2 = 2\, \Sigma\, q^{m^2/4};$$

whence, for b, c integers $\geqq 0$, we have, obviously

$$(15)\quad \vartheta_2^b(q^4) \vartheta_3^c(q^4) = \Sigma\, q^n [2^b N(n, b + c, b)].$$

For passing from representations to compositions, the following is useful:

$$(16)\quad \vartheta_2^b(q^4) \vartheta_3^c(q^4) = \Sigma\, q^n [N'(n, b + c, b)],$$

* The third of the relations (10) is from $\mathrm{sn}(kx, 1/k) = k\,\mathrm{sn}(x, k)$; and the fourth follows from this by actually forming the product $\mathrm{sn}u \times \mathrm{sn}u$, and noting that the coefficient is necessarily a reciprocal polynomial in k^2. It is important for the verification of Liouville's α_t coefficients to observe that $s_0' = s'_{r-1} = 2^{2r-1}$, which may be seen at once from $S_{2r}'(k^2)$, whose absolute term is $\frac{1}{2}[(1 + 1)^{2r} - (1 - 1)^{2r}]$.

where $N'(n, p, q)$ is the number of representations of n as a sum of p squares, the first q of which are odd with roots $\gtrless 0$, and the last $p - q$ even with roots $\gtrless 0$.

4. Equating coefficients of $(-1)^{r-1}u^{2r-1}/(2r-1)!$ in (5), we find, on using (11),

$$(17) \quad \frac{4}{k}\left(\frac{\pi}{2K}\right)^{2r} \Sigma\, q^{m/2}\zeta_{2r-1}(m) = S_{2r-1}(k^2) \equiv \sum_{t=0}^{r-1} s_t k^{2t},$$

which, on substituting for k, K their values from (14), becomes, after some obvious reductions,

$$(18) \quad 4\,\Sigma\, q^{m/2}\zeta_{2r-1}(m) = \sum_{t=0}^{r-1} s_t \vartheta_2^{4t+2}\vartheta_3^{4r-4t-2}.$$

From (6), in the same way,

$$(19) \quad 4\,\Sigma\, q^{m/2}\xi_{2r}(m) = \sum_{t=0}^{r-1} c_t \vartheta_2^{4t+2}\vartheta_3^{4r-4t}.$$

Changing q into q^4 in (18), (19), and applying (15), we get

$$(20) \quad \zeta_{2r-1}(m) = \sum_{t=0}^{r-1} 2^{4t} s_t N(2m, 4r, 4t+2);$$

$$(21) \quad \xi_{2r}(m) = \sum_{t=0}^{r-1} 2^{4t} c_t N(2m, 4r+2, 4t+2);$$

by equating the coefficients of q^{2m} in the respective series. To verify Liouville's statements about the coefficients A_t, B_t of § 2, we note that (20), (21) become respectively (1), (2) on putting $A_t = 2^{4t}s_t$, $B_t = 2^{4t}c_t$; hence by (10) the verification is complete. The calculation for (8) is similar to the foregoing. We find first, on using (13),

$$\frac{2}{k^2}\left(\frac{\pi}{K}\right)^{2r+2} \Sigma\, q^n \zeta'_{2r+1}(n) = \sum_{t=0}^{r-1} s_t' k^{2t};$$

which reduces to

$$(22) \quad 2^{2r+3}\,\Sigma\, q^n \zeta'_{2r+1}(n) = \sum_{t=0}^{r-1} s_t' \vartheta_2^{4t+4}\vartheta_3^{4r-4t};$$

and this, on expressing ζ' in terms of ζ, by (12), substituting q^4 for q, and using (16), becomes, for $n = 2^a m$,

$$(23) \quad 2^{a(2r+1)}\zeta_{2r+1}(m) = \sum_{t=0}^{r-1} 2^{4t-2r+1} s_t' N(4n, 4r+4, 4t+4).$$

Corresponding to (22), (7) gives, on substituting for D_{2r} its value in terms of C_{2r} from (9),

$$2\left(\frac{\pi}{K}\right)^{2r+1}\Sigma q^n\xi_{2r}''(n) = k^{2r}\sum_{t=0}^{r-1}c_t k^{-2t}; \tag{24}$$

whence

$$2^{2r+2}\Sigma q^n\xi_{2r}''(n) = \sum_{t=0}^{r-1}c_t\vartheta_2^{4r-4t}\vartheta_3^{4t+2}; \tag{25}$$

$$2^{2ar}\xi_{2r}(m) = \sum_{t=0}^{r-1}c_t 2^{2r-4t-2}N(4n,\ 4r+2,\ 4r-4t), \tag{26}$$

the last by changing q into q^4 in (25) and equating coefficients of q^{4n}; $n = 2^a m$, $a \geqq 0$. This may clearly be rewritten

$$2^{2ar}\xi_{2r}(m) = \sum_{t=0}^{r-1}2^{4t-2r+2}c_{r-t-1}N(4n,\ 4r+2,\ 4t+4). \tag{27}$$

Again, on writing $\alpha_t = 2^{4t-2r+1}s_t'$, $\beta_t = 2^{4t-2r+2}c_{r-t-1}$, we see by (10) that these coefficients have the properties announced by Liouville, and that (23), (27) are identical with (3), (4). The r-limitations $r \geqq 0$, etc., in (1) to (4) are obviously met.

5. The coefficients s, s' of (20), (23) may be calculated by recurrence, as shown by Gudermann.* The coefficients of (21), (27) are more easily found by Hermite's method.† But whichever method is used, the computation involves great labor if $r > 20$; and neither gives the c, s, s' as explicit functions of r, t. It would be desirable, for the consideration of several questions related to compositions as sums of squares, to have these functions of r, t explicitly. Thus, e.g., were they known, it would be possible, following Liouville's indications,‡ to state necessary and sufficient conditions that

* *Crelle*, vol. 19 (1839), pp. 79–83. Some of the coefficients in Gudermann's § 115 seem to be incorrect, but the essential part of his method on p. 79 is unaffected by this. Another recurrence for the s-coefficients is given by $\mathrm{sn}'u = \mathrm{cn}u\,\mathrm{dn}u$, and this one enables us to apply Hermite's method to the calculation of the s, s' as well as the c.

† "Rémarques sur le développement de cos *amx*;" *J. des Math.* (2), vol. 9 (1869), p. 289; *Paris C. R.*, vol. 77 (1863), p. 613. Apparently the quantities first denoted by $A_0, A_1, \cdots, A_n$ in Hermite's paper should each be multiplied by $(-1)^{n+1}$. In the rest of the paper they are taken with this meaning, so that the final results appear as intended. The misprint is of no importance for Hermite's purpose, but it is for ours; since otherwise the coefficients would not all be positive.

‡ *J. des Math.* (2), vol. 6, pp. 234–5. If (28)–(31) of the present paper are used in this connection, the powers of 2 do not appear as factors (as in Liouville's illustration), but only the binomial coefficients.

$N(2n, 2r)$ shall be a ξ, ζ-function of the divisors of n alone, for general r and n. The coefficients s, s', c, and hence A, B, α, β are clearly all integers > 0; but beyond this, and the few facts concerning them which were observed by Liouville, little if anything of arithmetical importance is known about them. The comparative largeness of their prime factors is significant.

6. It will be worth while, for its bearing on the question of composition, to write down the formulas which correspond to (1)–(4), when the restriction that the odd squares shall have positive roots is removed. They are, from (15), (16), (20), (21), (23), (27), as follows:

$$4\zeta_{2r-1}(m) = \sum_{t=0}^{r-1} s_t N'(2m, 4r, 4t+2), \tag{28}$$

$$4\xi_{2r}(m) = \sum_{t=0}^{r-1} c_t N'(2m, 4r+2, 4t+2), \tag{29}$$

$$2^{a(2r+1)+2r+3}\zeta_{2r-1}(m) = \sum_{t=0}^{r-1} s_t' N'(n, 4r+4, 4t+4), \tag{30}$$

$$2^{2(ar+r+1)}\xi_{2r}(m) = \sum_{t=0}^{r-1} c_{r-t-1} N'(n, 4r+2, 4t+4), \tag{31}$$

where m, n, r are as in § 2.

7. For $N(n, 2r)$ as defined in § 1, we have

$$N(n, 2r) = N'(n, 2r, 0) + \binom{2r}{1}N'(n, 2r, 1) + \binom{2r}{2}N'(n, 2r, 2) + \cdots + N'(n, 2r, 2r),$$

where $\binom{a}{b} = a!/b!(a-b)!$. Hence, it is easily seen from (28), a sufficient condition that $N(2m, 4r)$ shall be expressible in terms of the divisors of $2m$ alone is that the ratio $s_t : \binom{4r}{4t+2}$ shall have the same value for $t = 0, 1, 2, \cdots, r-1$. This condition is satisfied for $r = 3$. When $r = 4, 5, 6$, it is not. Or again, if some but not all of $s_1, s_2, \cdots, s_{r-2}$ are divisible by one prime $> 4r$, then clearly the condition is not satisfied. This criterion rejects $r = 5$, the prime factor 307 occurring in two out of the possible three terms; or this case is rejected by 83, which is a factor of only the middle term. Similar conclusions may be read from (29), (30), (31).

THE UNIVERSITY OF WASHINGTON.

SOME FUNCTIONAL EQUATIONS IN THE THEORY OF RELATIVITY.

BY PROFESSOR ARTHUR C. LUNN.

(Read before the American Mathematical Society April 26, 1919.)

IN transcribing the process of light-signalling which leads to his kinematic equations of transformation, Einstein* obtains relations, connecting the space-time coordinates in two systems of reference, in the form of functional equations. He solves these by passing to partial differential equations, and restricts the solutions to linearity by a rather obscure appeal to the homogeneity of time and space. In view of the fundamental character that his theory has proved to possess it is natural to ask more closely what is implied in the term homogeneity, which occurs in various senses in physical science, and whether it is necessary to assume differentiability. These and some related questions bearing on the fundamentals of that theory suggested the following analysis, which aims at a more complete transcription by means of functional equations and their solution without appeal to differentiation.

The general setting of the problem is found in the supposition that for the set of space-time points there are various systems of reference, corresponding to observers and carriers of coordinate frames in uniform translatory motion with respect to each other; these systems being, under conditions to be described, all on a parity with each other in the sense to be defined as the meaning of relativity. It is understood that every space-time point is identifiable by each system in terms of a unique quadruple of coordinates, and conversely that each such quadruple that occurs at all identifies a unique space-time point. For a physical theory some limitation on the range of the coordinates might perhaps seem appropriate, and for the most part even if such a limitation of any natural kind were imposed the results would probably be unchanged if interpreted for corresponding regions. But in order to avoid the complication of boundary conditions that would accompany such a limitation it will be supposed that every

* Einstein, *Drude Annalen*, vol. 17, pp. 891–921 (1905).

point of euclidean space of three dimensions exists at each instant for every observer, that no other points ever exist, and that the space-time manifold is in a certain impartial sense continuous. For analytic purposes this is taken to mean that identification of the same space-time points by any two observers yields a unique one-to-one continuous correspondence of their quadruples of coordinate values, existing for the entire infinite real range. In other words the coordinates (τ, ξ, η, ζ) in any system Σ are single-valued continuous functions of the coordinates (t, x, y, z) in any other system S, for all real values of the latter; these functions having then in some definite sense, though perhaps not algebraically, single-valued inverses. The problem is to determine the form of the functions by the imposition of conditions suitable to the notion of relativity as understood in the Einstein theory. For each system by itself the full kinematics of euclidean space and optical time is taken for granted.

Einstein's postulate of the constancy of the velocity of light has been much discussed, yet seems to invite still more detailed analysis than it has hitherto received. It clearly involves along with its physical content a considerable element of convention, with respect to orientation of axes and choice of units for instance. The essential physical feature seems to be that there is supposed to exist in every system by itself a cartesian scheme of measurement for space and time such that as interpreted in terms of that scheme space is optically isotropic; that is, light travels in straight lines with a velocity constant for each line and the same for all. This involves of course that much-discussed independence of the motion of the source of the light. For Einstein's plan of testing clocks by means of to-and-fro light signals, however, the rectilinearity of the light path is not directly made use of. It is in effect assumed that, starting at any instant, a light-signal can be sent from any point to any other, and that the distance is proportional to the time of transit. The factor of proportionality, the velocity of light, is assumed to be the same for all pairs of points, in any single system, and for convenience is taken as having the same numerical value c in every system by means of a single admissible restriction on the units of length and time in each. For the present purpose it will be assumed that in every system the corresponding scheme of measurement is of the kind described. The nature of the conditions

to be imposed is such that there remain arbitrary in each system the choice of origin, euclidean transformation of space-axes, and one further condition on the units.

That the system Σ as observed by S has a uniform translatory rigid motion means: that all points which in Σ appear fixed have in S a common velocity, of constant direction and of constant magnitude v, and conversely that any set of points having in S this particular velocity appear in Σ fixed. For consideration of a particular pair of systems let the direction of this velocity specify the direction of the x-axis in S. The light signals are to be thought of as passing between points fixed in Σ and described in S. A signal is longitudinal when these points are at one instant and therefore always on a line in S parallel to the x-axis; it is lateral when the points lie on a moving line perpendicular to the x-axis.

A forward longitudinal signal between points that in S are at a distance a apart may be described as starting at (t, x, y, z), going to

$$\left(t+\frac{a}{c-v},\ x+\frac{ca}{c-v},\ y,\ z\right),$$

then to

$$\left(t+\frac{2ca}{c^2-v^2},\ x+\frac{2cva}{c^2-v^2},\ y,\ z\right);$$

or more conveniently, by the use of the relative abscissa $x' = x - vt$, as involving successively

$$[t, x', y, z], \qquad \left[t+\frac{a}{c-v},\ x'+a,\ y,\ z\right],$$

and

$$\left[t+\frac{2ca}{c^2-v^2},\ x',\ y,\ z\right];$$

a backward longitudinal signal over an object of the same apparent length is like this except that (a, c) are replaced by $(-a, -c)$. Since the first and third are spatially coincident in Σ the intermediate τ must be the arithmetic mean of the others. Hence, if $\tau = F(x', t)$, where for brevity (y, z, v) are omitted temporarily from the functional symbol because only parameters, the conditions are the Einstein functional equation and its mate for opposite signals:

$$F(x', t) + F\left(x', t + \frac{2ca}{c^2 - v^2}\right)$$

(1)

$$= 2F\left(x' + a,\ t + \frac{a}{c - v}\right) = 2F\left(x' - a,\ t + \frac{a}{c + v}\right).$$

The combined or right hand equality says in effect that $F(\alpha, \beta)$ is invariant under the transformation

$$\alpha' = \alpha + 2a, \qquad \beta' = \beta + \frac{2va}{c^2 - v^2},$$

with a arbitrary, and is therefore a function of the invariant

$$\beta - \frac{v\alpha}{c^2 - v^2}$$

only, so that

$$F(x', t) = F\left(t - \frac{vx'}{c^2 - v^2}\right).$$

Substitution in the first equality in (1) reduces it to the form

$$F(u) + F(u + 2h) = 2F(u + h),$$

so that F, being continuous, is linear. Hence

$$\tau = \frac{\varphi(y, z, v)}{\kappa}\left(t - \frac{vx}{c^2}\right) + f(y, z, v), \tag{2}$$

where

$$\kappa^2 = 1 - \frac{v^2}{c^2}.$$

Here the first coefficient is written in a form later convenient, the φ corresponding to Einstein's.

A lateral signal over an object of width b in a plane of constant z would involve

$$(t, x, y, z), \qquad \left(t + \frac{b}{c\kappa},\ x + \frac{vb}{c\kappa},\ y + b,\ z\right),$$

and

$$\left(t + \frac{2b}{c\kappa},\ x + \frac{2vb}{c\kappa},\ y,\ z\right).$$

The condition that here also the intermediate τ be the arithmetic mean reduces to

$$\left[t-\frac{vx}{c^2}+\frac{b}{c}\right][\varphi(y, z, v)-\varphi(y+b, z, v)]$$
$$+[f(y, z, v)-f(y+b, z, v)]=0,$$

showing that φ and f must be independent of y and by symmetry of z also.

If the lateral signal be considered first, and $\tau = F(x', y, t)$, then the corresponding condition is

$$F(x', y, t)+F\left(x', y, t+\frac{2b}{c\kappa}\right)$$
$$(3)\qquad =2F\left(x', y+b, t+\frac{b}{c\kappa}\right)=2F\left(x', y-b, t+\frac{b}{c\kappa}\right),$$

which shows that τ must be independent of y, and similarly of z, and must be linear in t, so that

$$(4)\qquad \tau=\psi(x', v)t+g(x', v).$$

Then the longitudinal signal gives the condition

$$t[\psi(x', v)-\psi(x'+a, v)]$$
$$+\frac{a}{c-v}\left[\frac{c}{c+v}\psi(x', v)-\psi(x'+a, v)\right]$$
$$+[g(x', v)-g(x'+a, v)]=0$$

which shows that ψ must be independent of x' and that

$$(5)\qquad g(x'+a, v)-g(x', v)=-\frac{av}{c^2-v^2}\psi(v),$$

so that $g(x', v)+vx'\psi(v)/(c^2-v^2)$ must be a function of v only. The two orders of proof thus lead to the result

$$(6)\qquad \tau=\frac{\varphi(v)}{\kappa}\left(t-\frac{vx}{c^2}\right)+f(v),$$

which evidently contains all the information that the signalling process can give about the time reckoning in Σ, since the undetermined functions f and φ correspond to arbitrary choice of origin and units. This equation shows that the time intervals in Σ of one-way transit are $a\varphi(v)/\kappa c$ and $b\varphi(v)/c$ for the longitudinal and lateral signals respectively.

The condition that in each of the instances mentioned of

to-and-fro signal the initial and final space points coincide in Σ shows that (ξ, η, ζ) are functions of (x', y, z, v) only, independent of t; which is in fact equivalent to the condition of rigid motion of Σ with respect to S, when the time reckoning in Σ is given by (6). The relation between the two systems of space coordinates will appear from consideration of a one-way signal of arbitrary direction and time interval. For convenience let $x'/\kappa = x''$; then in general

$$\Delta x = \kappa \Delta x'' + v\Delta t, \qquad \Delta\tau = \varphi(v)(\kappa\Delta t - v\Delta x''/c^2).$$

From these it follows that the distance r, which must be the same as $c\Delta t$, travelled in the time interval Δt by the signal in (x, y, z) space, is related to the apparent distance r'' in the space of the auxiliary coordinates (x'', y, z) by

$$r^2 = c^2(\Delta t)^2 = r''^2 - v^2(\Delta x'')^2/c^2 + v^2(\Delta t)^2 + 2\kappa v\Delta x''\Delta t,$$

which reduces to

$$r''^2 = (\kappa c\Delta t - v\Delta x'')^2 = c^2(\Delta\tau)^2/\varphi^2.$$

This shows that the distance in (ξ, η, ζ) space is always the same as in $(\varphi x'', \varphi y, \varphi z)$ space, so that these two triples must be linear orthogonal transforms of each other; the known proof of this requires no assumption of differentiability. Combined with (6) this gives essentially Einstein's result, except that there is here no initial supposition as to the orientation of the axes in Σ.

If, however, the special supposition be made that ξ is a function of (x, t) only, therefore of (x', t) only, then the longitudinal signal gives

$$G\left(x', t + \frac{2ca}{c^2 - v^2}\right) = G(x', t), \tag{7}$$

making G independent of t; also

$$G(x' + a) = G(x') + a\varphi(v)/\kappa, \tag{8}$$

so that, G being continuous, $\xi = \varphi(v)x'/\kappa + g(v)$. Then the remaining conditions show that the pair (η, ζ) is an orthogonal transform of $(\varphi y, \varphi z)$. The general result is conveniently viewed from Minkowski's point of view, as giving $(ic\tau, \xi, \eta, \zeta)$ from (ict, x, y, z) by a rigid rotation about an arbitrary origin, followed by a magnification with factor φ, whose arbitrariness indicates the freedom remaining in the choice of units.

Thus far the proof refers to a particular pair of systems only, and the undetermined functions φ, f, g, and the like, instead of being written as functions of v, are thus more suitably to be regarded as merely functions of the pair of systems whose relation is sought. For there is nothing in the general assumptions to imply that their values need be the same for two Σ's moving with respect to S with the same velocity, even if the directions were also the same, in which case they would be relatively to each other at rest. But in the case of a set of systems, each standing to each in the relation restricted by the general postulates, certain conditions are imposed on these functions by the requirement of transitivity, or the group property as it here becomes. For example, with n systems there would be $n(n-1)/2$ factors φ and their inverses, but only $(n-1)$ arbitrary ratios of units. If then instead of $\varphi(v)$ there be written $\varphi(\Sigma, S)$, which for this purpose might be more suggestively $\varphi(\Sigma/S)$, these conditions have the form

$$\varphi(U, T)\varphi(T, S) = \varphi(U, S). \tag{9}$$

The algebraic form of the implications of this equation would depend on the set of systems considered. If the artificiality be discarded that would allow φ to be different for two Σ's at rest with respect to each other or moving with respect to S with the same speed in different directions, the condition reduces to

$$\varphi(v)\varphi(v') = \varphi(v''), \tag{10}$$

where v'' is the resultant of v and v' in Einstein's sense. If any rigid translation with respect to any S gives a possible Σ, then, by alteration of the direction of v', v'' can vary while v and v' are constant, so that φ must be constant and therefore unity, as Einstein makes it. Poincaré* had already indicated this as a condition for a group in the Lorentz transformation.† If, however, only parallel motions were admitted, and v taken as with sign, the condition would be

$$\varphi(v)\varphi(v') = \varphi\left(\frac{v+v'}{1+vv'/c^2}\right). \tag{11}$$

The continuous solution of this, readily obtained through the transformation $v/c = \tanh \alpha$, is

* Poincaré, *Comptes Rendus*, vol. 140, pp. 1504–1508 (1905).

† Lorentz, *Proc. Amsterdam A.*, vol. 6, pp. 809–831 (1904); and Theory of Electrons, p. 197.

$$\varphi(v) = \left(\frac{c+v}{c-v}\right)^n, \tag{12}$$

where n is an arbitrary constant. Einstein's condition that reversal of v leave φ unaltered makes $n = 0$; as to physical interpretation it may be noted that this is the only case where the contraction factor would have an expansion in powers of v free from a linear term.

Einstein's original proof referred only to signals transmitted through a vacuum. The case of a refracting medium of arbitrary index μ, at rest with respect to Σ, was considered by Laub,* who imposed the requirement that the comparison of space and time reckoning be independent of the value of μ. He obtained thereby the various velocities of propagation with respect to S, leading to the Fresnel-Fizeau convection coefficients. These results were then shown by Laue† to be instances of the Einstein law of composition of velocities. The following examples illustrate the modification of the functional equations to suit this case.

The medium is supposed to appear isotropic in Σ, with velocity of propagation c/μ; but in S light will have different velocities in different directions, except when μ is unity. Considering only the longitudinal signal let the forward and backward velocities in S be c', c''. Then (1) is replaced by

$$\begin{aligned} &F(x', t) + F\left(x', t + a\left[\frac{1}{c'-v} + \frac{1}{c''+v}\right]\right) \\ &\quad = F\left(x'+a, t+\frac{a}{c'-v}\right) \\ &\quad = F\left(x'-a, t+\frac{a}{c''+v}\right), \end{aligned} \tag{13}$$

making $F(\alpha, \beta)$ invariant under the transformation

$$\alpha' = \alpha + 2a, \qquad \beta' = \beta + a\left[\frac{1}{c'-v} - \frac{1}{c''+v}\right],$$

so that

$$F(x', t) = F\left(t - \frac{x'}{2}\left[\frac{1}{c'-v} - \frac{1}{c''+v}\right]\right).$$

Linearity appearing as before, the result is

* Laub, *Ann. d. Phys.*, vol. 23, pp. 738–744 (1907).
† Laue, *Ann. d. Phys.*, vol. 23, pp. 989–990 (1907).

$$(14)\quad \tau = \frac{\varphi(v,\mu)}{\kappa}\left(t - \frac{x'}{2}\left[\frac{1}{c'-v} - \frac{1}{c''+v}\right]\right) + f(v,\mu).$$

Equation (7) is replaced by

$$(15)\quad G(x',t) = G\left(x',\ t + a\left[\frac{1}{c'-v} + \frac{1}{c''+v}\right]\right),$$

the equation (8) by

$$(16)\quad G(x'+a) = G(x') + \frac{c\varphi a}{2\mu\kappa}\left[\frac{1}{c'-v} + \frac{1}{c''+v}\right],$$

whence

$$(17)\quad \xi = \frac{c\varphi x'}{2\mu}\left[\frac{1}{c'-v} + \frac{1}{c''+v}\right] + g(x,\mu).$$

The corresponding modifications for lateral and other signals are readily made.

It appears therefore that Einstein's assumption of differentiability is unnecessary, and that the needed features of "homogeneity" are already implicit in the optical and kinematic postulates as here interpreted.

THE UNIVERSITY OF CHICAGO,
April 8, 1919.

FORMULAS FOR CONSTRUCTING ABRIDGED MORTALITY TABLES FOR DECENNIAL AGES.

BY PROFESSOR C. H. FORSYTH.

MR. GEORGE KING presented in the British Registrar-General's Report for 1914 a method of constructing abridged mortality tables which consist merely of the values appearing in an ordinary mortality table but corresponding only to each quinquennial age. The computation of the values corresponding to the other ages is eliminated by the method and an enormous saving of labor is thus gained.

Such tables promise to prove very useful in investigations in human vitality because many problems which have heretofore involved too much computation to permit much investigation now become relatively easy. For this reason it seems

well worth while to derive and present the analogous formulas for constructing mortality tables for *decennial* ages.

Tables based on decennial ages require little more than half the computation necessary for tables based on quinquennial ages after the proper statistical data are collected and properly arranged,—the computation takes little over an hour even when the incidental computation is performed by hand—give practically the same information and involve much more suitable ages (i.e., ages 10, 20, 30, etc., instead of 12, 17, 22, etc.).

First, the given population statistics and the mortality statistics are each easily arranged in the age groups 5–14,* 15–24, 25–34, etc. Then, the following formula† to second differences‡ used for interpolating ordinates among areas

$$u_{x/t} = w_0 + (2x - t + 1)\frac{\Delta w_0}{2t^2} + \{3x^2 + 3x(1 - 2t) + (1 - 3t + 2t^2)\}\frac{\Delta^2 w_0}{6t^3}, \tag{1}$$

where

$$w_{x/t} = u_{x/t} + u_{(x+1)/t} + \cdots + u_{(x+t-1)/t}$$

and $w_{nx t/t}$ is generally abbreviated to w_{nx}, is applied to determine the population and the deaths for a suitable individual age. Thus, if w_0, w_1, and w_2 refer to the populations (or deaths) for the age groups 5–14, 15–24, and 25–34 respectively, $u_{x/t}$ for $t = 10$ and $x = 15$ gives the population (or deaths) for the individual age 20, and

$$\begin{aligned} u_{15/10} &= .1(w_0 + \Delta w_0) + \frac{.01}{2}(\Delta w_0 - .3\Delta^2 w_0) \\ &= .1w_1 + \frac{.01}{2}(\Delta w_0 - .3\Delta^2 w_0) \end{aligned} \tag{2}$$

and similarly for ages 30, 40, etc.

Knowing the population L_x and the number of deaths d_x at an individual age x, the value of the probability of living one year or p_x is obtained by the well-known formula

* The great variation and uncertainties of the death rate at ages in the neighborhood of birth necessitate the omission of the age group 0–5.

† *Quarterly Publications of the American Statistical Association*, Dec. 1916.

‡ It is generally accepted that second differences are sufficient when dealing with ordinary statistical data.

$$(3) \qquad p_x = 1 - q_x = \frac{L_x - \frac{1}{2}d_x}{L_x + \frac{1}{2}d_x}.$$

Probably, a better plan for computing values of p_x is to combine formulas (2) and (3) to give

$$p_{x+5} = \frac{.1(W_x - w_x) + \frac{.01}{2}\{(\Delta W_{x-10} - \Delta w_{x-10}) - .3(\Delta^2 W_{x-10} - \Delta^2 w_{x-10})\}}{.1(W_x + w_x) + \frac{.01}{2}\{(\Delta W_{x-10} + \Delta w_{x-10}) - .3(\Delta^2 W_{x-10} + \Delta^2 w_{x-10})\}},$$

where W_x and w_x refer to the population and *one-half* the deaths respectively, corresponding to the decennial age group running from x to $x + 9$ and the subscripts are changed to a more workable form to correspond to the ages. If we write D_x for $W_x - w_x$ and S_x for $W_x + w_x$ and notice that

$$\Delta^n W_x \pm \Delta^n w_x = \Delta^n (W_x \pm w_x) = \Delta^n S_x \qquad \text{or} \qquad \Delta^n D_x$$

the formula given above reduces to

$$(4) \qquad p_{x+5} = \frac{D_x + \frac{.1}{2}(\Delta D_{x-10} - .3\Delta^2 D_{x-10})}{S_x + \frac{.1}{2}(\Delta S_{x-10} - .3\Delta^2 S_{x-10})}.$$

If we let $t = 10$ and $x = 5$ in formula (1), we obtain in like manner

$$(4') \qquad p'_{x+5} = \frac{D_x + \frac{.1}{2}\Delta D_x - .06\frac{1}{2}\Delta^2 D_x}{S_x + \frac{.1}{2}\Delta S_x - .06\frac{1}{2}\Delta^2 S_x},$$

which may be applied to the age groups 5–14, 15–24, and 25–34 to give p_{10}.

To apply formulas (4) and (4′), columns of values of D_x and S_x are first obtained by inspection from columns of values of W_x and w_x; the divisions are then performed by logarithms to give a column of values of $\log p_x$.

Values of $\log {}_{10}p_x$ are next obtained from values of $\log p_x$ as follows: If we sum the ten expressions obtained by sub-

stituting 10/10, 11/10, ···, 19/10 in Newton's formula (to second differences)

$$u_x = u_0 + x\Delta u_0 + \frac{x(x-1)}{2}\Delta^2 u_0,$$

we obtain

$$w_1 = 10u_0 + 14\tfrac{1}{2}\Delta u_0 + 3\tfrac{2}{3}\Delta^2 u_0,$$

which may be written

$$w_1 = 10u_1 + 4\tfrac{1}{2}\Delta u_0 + 3\tfrac{2}{3}\Delta^2 u_0, \tag{5i}$$

where, however, the true coefficient of $\Delta^2 u_0$ is 3.675. The arbitrary change in coefficients simplifies the computation considerably while the error thus introduced will generally prove insignificant in the final results.

Since $\log {}_{10}p_x = \log p_x + \log p_{x+1} + \cdots + \log p_{x+9}$, formula (5*i*) enables one to obtain a column of values $\log {}_{10}p_{20}$, $\log {}_{10}p_{30}$, etc., from a column of values $\log p_{10}$, $\log p_{20}$, etc. The value of $\log {}_{10}p_{10}$ is obtained by using the formula

$$w_0 = 10u_0 + 4\tfrac{1}{2}\Delta u_0 + .8\tfrac{1}{4}\Delta^2 u_0, \tag{5i'}$$

which is the sum of the ten expressions obtained by substituting 0/10, 1/10, ···, 9/10 in Newton's formula.

Assuming a suitable radix or number of individuals living at age 10, the values of $\log {}_{10}p_x$ are added accumulatively to the logarithm of the radix to give in succession the logarithm of the number of survivors at each subsequent tenth age.

The radix is usually assumed to be 100,000 at age 10 but after considerable experience with the difficulties encountered in trying to complete such mortality tables at the higher ages the writer has become convinced that the radix of at least an *abridged* mortality table should not ordinarily be greater than 1,000 at age 10. Even then in a few exceptional cases it has been found necessary to adopt rather arbitrary measures to complete the tables at the higher ages. As the values corresponding to the highest ages have very little effect on the values at the earlier ages, the adoption of a radix of 1,000 obviates the necessity of introducing arbitrary and often questionable measures *for unimportant details.*

The column of survivors for decennial ages obtained in the manner just explained constitutes the abridged mortality table desired, but a column of expectations of life is always desirable and is obtained as follows. If we sum the expressions obtained by substituting 11/10, 12/10, ···, 20/10 in Newton's formula, we get

$$(5t) \qquad w_{11/10} = 10u_1 + 6\tfrac{1}{2}\Delta u_0 + 4\tfrac{2}{3}\Delta^2 u_0,$$

where the coefficient $4\frac{2}{3}$ is written instead of the true value 4.675 to simplify computation.

If formula (5*t*) is applied to the column of survivors l_{10}, l_{20}, etc., constituting the abridged mortality table, we obtain the number of survivors corresponding to the age groups 20–29, 30–39, etc. The number of survivors corresponding to the age group 10–19 is obtained by applying the formula

$$(5t') \qquad w_{1/10} = 10u_0 + 5\tfrac{1}{2}\Delta u_0 + .8\tfrac{1}{4}\Delta^2 u_0,$$

which is the sum of the expressions obtained by substituting 1/10, 2/10, $\cdots$, 10/10 in Newton's formula.

By way of distinction, formulas (5*i*) and (5*i'*) have been called *initial* forms and formulas (5*t*) and (5*t'*) *terminal* forms, for obvious reasons.

If, now, the column of survivors for the decennial age groups be summed accumulatively, beginning at the highest ages and the successive sums be divided by the number of survivors at the individual age just preceding in each case—as given in the abridged mortality table—a column of values of *curtate* expectation of life is obtained. Assuming, however, that the average individual lives a half year in the year of his death, it is customary to add 0.5 to each of the values just mentioned to obtain values of the *complete* expectation of life—the values usually quoted as the expectation of life.

In computing the expectation of life there is no satisfactory way of determining the number of survivors corresponding to the last age group. This fact should give no serious concern, for beyond much doubt the errors in the data at the higher ages—especially those due to the consistent overstatement of ages—render worthless any special efforts to be accurate at these ages. If only one decimal place is kept in the values of the expectation of life, it will make no difference at all what the number of survivors corresponding to the last age group is taken to be, because it will not affect any of the values of the expectation of life except those at the very highest—say, two or three—decennial ages and these values because of their inaccuracy should not be given very much weight.

The values of the death rates q_x are obtained from the values of p_x by means of the relation between q_x and p_x given in formula (3).

DARTMOUTH COLLEGE.

SHORTER NOTICES.

Projective Geometry. By L. WAYLAND DOWLING. McGraw Hill Book Company, 1917. xiii + 215 pp.

THIS book (as the preface tells us) has been developed from a course of lectures given by the author for a number of years at the University of Wisconsin.

In its treatment of the subject it follows closely the main lines of the classical text of Reye in nomenclature and in point of view. It begins with an introductory chapter giving definitions of the elements, projection, section, and ideal elements. The point range, sheaf of lines, sheaf of planes, field of points, field of lines, bundle of lines, bundle of planes, space of points, space of planes, special linear complex, and space of lines are named as the eleven primitive forms. No formal definition of primitive form is given which would enable the reader to judge why exactly these eleven forms are given and others are excluded.

The concept of motion of a line about a point and that of parallelism are made use of in defining an ideal point, and this is followed by the "fundamental assumption: On every straight line there is one and only one ideal or infinitely distant point. This point makes the line continuous from any one point on it to any other point on it in either direction. Through a given point there can be drawn one and only one line parallel to a given line. This parallel intersects the given line in the ideal or infinitely distant point."

This quotation perhaps shows the point of view of the author in basing his projective geometry on a completely developed euclidean geometry. Duality is introduced by means of illustrations followed by a statement of the method of obtaining new theorems from others by interchanging elements and their reciprocals in the statements. Neither the logical bearing of the principle nor the extent of its validity is discussed. The study of the complete quadrangle leads up to harmonic forms and the cross-ratio of a harmonic range is shown to be -1 by the use of ordinary metrics. From harmonic ranges harmonic scales are developed. The harmonic separation theorems are obtained from a free and intuitive use of continuity. A form of the Dedekind postulate is

introduced using the concepts of segments and order as modified by the introduction of ideal elements. The proof of the fundamental theorem of von Staudt is then based on continuity. Curves and envelopes of the second order, poles and polars with respect to a curve are then studied in considerable detail, with the use of metrics for some properties.

An involution on a form is defined as a cyclic projectivity of order 2 and conjugate imaginary points, lines and planes are introduced from the elliptic involution. An entire chapter is devoted to the focal properties of conics. Collineations, dualities, affinities, and polarities are treated briefly.

The figures are fairly well done, the typography is very good, and the volume as a whole is neat and attractive. The large number of exercises scattered throughout the book add much to its utility as a textbook. The little historical notes although very brief are also valuable and stimulating.

In the opinion of the reviewer, such a book as this one, which is avowedly not concerned with the more logical phases of the subject, would be more valuable to many if it gave at least references to such books as that of Veblen and Young, where such treatment could be found. On the whole, however, the book will no doubt be of much service in beginning courses in the subject.

F. W. OWENS.

Introduction to the Elementary Functions. By RAYMOND BENEDICT MCCLENON, with the editorial cooperation of WILLIAM JAMES RUSK. Boston, Ginn and Company, 1918. x + 244 pp.

UNDER the above title the authors present in book form their idea of the content of a required course in mathematics for freshmen in a small college. As presented, the text is the result of five years of teaching of the course in such a college. The elementary functions are those from plane trigonometry, plane analytic geometry, and the elements of differential calculus. These subjects are combined with such review topics from algebra as is necessary to bind the topics presented into an organic whole. The chapters are not sections taken from a single one of the principal subjects, but are mixtures. The calculus covers only the differentiation of algebraic functions with applications to rate and maxima and minima problems.

The treatment of topics is informal and inductive, there being a noticeable absence of formal proofs. Naturally the reviewer does not agree with the arrangement of material nor with its content, but we do agree with an avowed object of the book, namely that of coaxing students in a small college, or elsewhere for that matter, into a further study of non-required mathematics. We object to the form of the discussion of the quadratic in one unknown (pages 46, 47) wherein the authors seem to discard complex roots, as *roots*. Of course problems in analytic geometry and elsewhere lead to equations with complex roots, which are not interpretable in terms of the real elements presented in the problem. But this does not seem to be the meaning of the authors.

C. F. CRAIG.

NOTES.

THE following mathematical papers have appeared in recent numbers of the *Proceedings of the National Academy of Sciences*: volume 5, number 1 (January, 1919): "A theorem in power series, with an application to conformal mapping," by T. H. GRONWALL; number 3 (March): "Tables of the zonal spherical harmonic of the second kind $Q_1(z)$ and $Q_1'(z)$," by A. G. WEBSTER and W. FISHER; number 4 (April): "On the real folds of abelian varieties," by S. LEFSCHETZ; "Covariants of binary modular groups," by O. E. GLENN; "The general solution of the indeterminate equation: $Ax + By + Cz + \ldots = r$," by D. N. LEHMER; number 6 (June): "On the most general class L of Fréchet in which the Heine-Borel-Lebesgue theorem holds true," by R. L. MOORE; "On a certain class of rational ruled surfaces," by A. EMCH; number 7 (July): "On the twist in conformal mapping," by T. H. GRONWALL; "Groups involving only two operators which are squares," by G. A. MILLER; "Real hypersurfaces contained in abelian varieties," by S. LEFSCHETZ.

THE concluding (June) number of volume 20 of the *Annals of Mathematics* contains the following papers: "Relations between abstract group properties and substitution groups," by G. A. MILLER; "The complete quadrilateral," by J. W.

CLAWSON; "Triply conjugate systems with equal point invariants," by L. P. EISENHART; "On a system of linear partial differential equations of the hyperbolic type," by T. H. GRONWALL; "Some properties of circles and related conics," by J. H. WEAVER; "Integrals in an infinite number of dimensions," by P. J. DANIELL; "On the differentiability of the solution of a differential equation with respect to a parameter," by J. F. RITT; "Note on the derivatives with respect to a parameter of the solutions of a system of differential equations," by T. H. GRONWALL; "On quaternions and their generalization and the history of the eight-square theorem. Addenda," by L. E. DICKSON.

The July number (volume 41, number 3) of the *American Journal of Mathematics* contains the following papers: "Invariants of differential geometry by the use of vector forms," by C. D. RICE; "On certain saltus equations," by HENRY BLUMBERG; "Investigations on the plane quartic," by TERESA COHEN; "On surfaces containing two pencils of cubic curves," by C. H. SISAM; "Modular invariants of a quadratic form for a prime power modulus," by J. E. MCATEE.

ON November 2, 1918, the Norwegian mathematical society was founded at Christiania. Its officers are Professors STÖRMER, president, BIRKELAND, vice-president, PALMSTRÖM, secretary, and SOLBERG, treasurer. The Society will publish a *Bulletin*, under the editorship of Professors Alexander and Heegaard.

THE firm of Julius Springer Berlin, announces the publication of a new journal devoted exclusively to original mathematical memoirs, the *Mathematische Zeitschrift*. It is edited by Professor L. Lichtenstein, with the collaboration of Professors K. Knopp, E. Schmidt, and I. Schur and an editorial committee consisting of Professors W. Blaschke, L. Féjer, C. Herglotz, A. Kneser, E. Landau, O. Perron, F. Schur, E. Study and H. Weyl. Two volumes appear annually.

ON December 12, 1918, Professor FELIX KLEIN celebrated the fiftieth anniversary of his doctorate, and on April 25, 1919, his seventieth birthday.

PROFESSOR C. KOSTKA, of Insterburg, Dr. W. LIETZMANN, of

Jena, Professor H. E. TIMERDING, of the Braunschweig technical school, and Professor W. LOREY, of Leipzig, have been elected members of the Leopoldinisch-Carolinische Akademie der Naturforscher, of Halle.

PROFESSOR E. GRÜNEISEN has been appointed associate professor of mathematical physics at the University of Marburg.

PROFESSOR G. HESSENBERG, of the Breslau technical school, has been made Geheimer Regierungsrat.

PROFESSOR ERHARD SCHMIDT, of the University of Berlin, has been elected a member of the mathematico-physical class of the academy of sciences of Berlin.

DR. A. TIMPE, of the University of Münster, has been appo.nted professor of mathematics at the agricultural school of Berlin.

PROFESSOR N. E. NÖRLUND, of the University of Lund, has been appointed professor of mathematics at the University of Copenhagen.

PROFESSORS W. FEUSSNER, of the University of Marburg, and G. FREGE, of the University of Jena, have retired from active teaching.

DR. E. JACOBSTHAL, of the Berlin technical school, has been promoted to a professorship of mathematics.

PROFESSOR H. VON MANGOLDT has been elected president of the Deutsche Mathematiker-Vereinigung for 1918–19. Professor FELIX KLEIN has been elected honorary president for the same period, in honor of the fiftieth anniversary of his doctorate.

PROFESSOR H. FEHR, of the University of Geneva, has been elected foreign correspondent of the Royal society of sciences of Liège.

DR A. MOHRMANN has been appointed professor of mathematics at the University of Basle.

PROFESSOR P. APPELL has retired from the office of dean of the faculty of sciences of the University of Paris, after sixteen years of service.

PROFESSOR H. ANDOYER, of the Sorbonne, has been elected member of the Paris academy of sciences, in the section of astronomy.

DR. H. LEBESGUE, of the University of Paris, has been promoted to a professorship of the application of analysis to geometry. Professor Lebesgue has been elected president of the Société mathématique de France for the year 1919.

DR. E. VESSIOT has been appointed professor of general mathematics at the University of Paris, as successor to Professor C. GUICHARD, who has been transferred to a professorship of higher geometry.

DR. J. CHAZY has been appointed professor of mathematics at the University of Lille, as successor to the late Professor J. CLAIRIN.

DR. P. HUMBERT has been apointed maître de conférences at the University of Montpellier, during the absence of Dr. A. DENJOY, who is at the University of Utrecht.

DR. L. ROY has been appointed professor of analytical mechanics at the University of Toulouse, as successor to the late Professor S. LATTÈS.

DR. H. R. HASSÉ, of the University of Manchester, has been appointed professor of mathematics at the University of Bristol.

AT the May meeting of the American academy of arts and sciences, Professors JOSEPH LIPKA, G. A. MILLER, F. R. MOULTON, and VIRGIL SNYDER were elected fellows in the section of mathematics and astronomy.

MAJOR E. V. HUNTINGTON, who has been assigned to statistical duty in Washington, D. C., during the past year, was recently made a member of the General Staff. He is returning to Harvard University in the autumn.

DR. HENRY BLUMBERG, of the University of Illinois, has been promoted to an assistant professorship of mathematics.

ASSISTANT professor W. W. RANKIN, of the University of North Carolina, has been granted leave of absence for the year 1919–20 and has been appointed instructor in mathematics at Columbia University for that period.

PROFESSOR M. E. GRABER, of Heidelberg University, Tiffin, Ohio, has been appointed professor of physics at Morningside College, Sioux City, Iowa.

ASSISTANT professor M. O. TRIPP, of the University of Maine, has been promoted to an associate professorship of mathematics.

PROFESSOR W. GROSS, of the University of Vienna, died October 29, 1918, at the age of thirty-two years.

THE death is reported of Dr. J. KELLER, of the technical school at Zürich.

PROFESSOR C. BRANDENBERGER, of the cantonal school at Zürich, died January 2, 1919.

PROFESSOR PAUL MANSION, of the University of Ghent, died April 18, 1919, at the age of seventy-five years.

PROFESSOR C. ALASIA DE QUESADA, of the gymnasium at Albengo, died November 19, 1918, at the age of forty-nine years.

PROFESSOR EMILE DUMONT, of the Institut Michot-Montgenast at Brussels, was killed in action in the late war.

THE Rt. Hon. JOHN WILLIAM STRUTT, Lord Rayleigh, died July 1, 1919, at the age of seventy-six years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BORTOLOTTI (E.). Italiani scopritori e promotori di teorie algebriche. Modena, O. Ferraguti, 1919. 8vo. 102 pp.

BOUMAN (P. J.) en ZELM (J. C. VAN). De rekenkundige denkbaarheden in logischen samenhang, met, als proeve van toegepaste logica, een rekenmethode voor de lagere school. Amsterdam, Versluys, 1918. 8vo. 381 pp. Fl. 2.90

BOUTROUX (P.). Les principes de l'analyse mathématique. Exposé historique et critique. Tome 2. Paris, Hermann, 1919. 8vo. 512 pp. Fr. 20.00

BRENDEL (M.). See KLEIN (F.).

CAJORI (F.). A history of mathematics. Second edition revised and enlarged. New York, Macmillan, 1919. 8vo. 10 + 514 pp. $4.00

KLEIN (F.), BRENDEL (M.) und SCHLESINGER (L.). Materialen für eine wissenschaftliche Biographie von Gauss. Heft 6: Die Wechselwirkung zwischen Zahlenrechnen und Zahlentheorie bei C. F. Gauss, von Ph. Maennchen. Leipzig, Teubner, 1918. 8vo. 47 pp. M. 3.00

LENGAUER (J.). Analytische Geometrie. 2te Auflage. Kempten, Kösel, 1918. 113 pp.

MAENNCHEN (P.). See KLEIN (F.).

MANNOURY (G.). Over de sociale betekenis van de wiskundige denkvorm. Groningen, Noordhoff, 1917. 8vo. 23 pp. Fl. 0.60

OPPERMANN (A.). Premiers éléments d'une théorie du quadrilatère complet. Paris, Gauthier-Villars, 1919. 8vo. 76 pp. Fr. 4.00

SCHLESINGER (L.). See KLEIN (F.).

SCHUH (F.). Leerboek der theoretische rekenkunde. Eerste deel. Groningen, Noordhoff, 1919. 8vo. 452 pp. Fl. 8.50

TEIXEIRA DE MATTOS (J.). Over satellietruimten en eenige toepassingen op satellietopperplakken. (Diss.) Amsterdam, H. G. van Dorsten. 8vo. 8 + 125 pp.

VOLLENHOVE (D. H. T.). De wijsbegeerte der wiskunde van theistisch standpunkt. (Diss.) Amsterdam, Wed. G. van Sost, 1918. 8vo. 15 + 444 pp.

ZELM (J. C. VAN). See BOUMAN (P. J.).

II. ELEMENTARY MATHEMATICS.

INGOLD (L.). See KENYON (A. M.).

KARPINSKI (L. C.). Four-place logarithmic and trigonometric tables, together with interest tables. Ann Arbor, George Wahr, 1918. 30 pp. Paper. $0.30

KENYON (A. M.) and INGOLD (L.). Elements of plane trigonometry, with brief tables. New York, Macmillan, 1919. 12mo. 9 + 117 + 18 + 12 pp. $1.00

———. Elements of plane trigonometry, with complete tables. New York, Macmillan, 1919. 12mo. 9 + 117 + 18 + 124 pp. $1.20

LENNES (N. J.). See SLAUGHT (H. E.).

LIETZMANN (W.). See SCHUSTER (M.).

LIETZMANN (W.) und ZÜHLKE (P.). Aufgabensammlung und Leitfaden für Arithmetik, Algebra und Analysis. Ausgabe B für Realanstalten, Oberstufe. Leipzig, Teubner, 1918. 334 pp.

———. Geometrische Aufgabensammlung. Ausgabe A für Gymnasien, Oberstufe. Leipzig, Teubner, 1918. 150 pp.

LIGHTFOOT (J.). An elementary algebra for junior students. London, Russell, 1919. 3s. 6d.

———. Graphic algebra for elementary and intermediate students. London, Russell, 1919. 1s.

MYERS (G. W.). See PALMER (C. I.).

PALMER (C. I.) and TAYLOR (D. P.). Solid geometry. Edited by G. W. Myers. Chicago, Scott, Foresman and Company, 1918. 6 + 178 pp.

ROBERTSON (J.). Dictionary for international commercial quotations to translate units of value from one currency, weight and measure direct into another. London, Oxford University Press, 1918. 8vo. 12 + 208 pp.

SCHUSTER (M.). Geometrische Aufgaben, herausgegeben von W. Lietzmann. Leipzig, Teubner, 1918. Ausgabe A (Vollanstalten), 3ter Teil: Stereometrie. 3te Auflage. Ausgabe B (Progymnasien und Realschulen): Planimetrie. 4te Auflage. 103 + 118 pp. Geb. M. 2.60 + 2.20

SLAUGHT (H. E.) and LENNES (N. J.). Solid geometry with problems and applications. Revised edition. Boston, Allyn and Bacon, 1919. 6 + 211 pp. $1.00

TAYLOR (D. P.). See PALMER (C. I.).

WATSON (B. M.) and WHITE (C. E.). Modern arithmetic. New York, Heath, 1918. Primary, 8 + 252 pp. Intermediate, 10 + 254 pp. For upper grades, 9 + 302 pp.

WHITE (C. E.). See WATSON (B. M.).

YOUNG (J. W. A.). The teaching of mathematics in the elementary and secondary school. New Impression. New York, Longmans, 1919. 8vo. 8 + 491 pp. $1.75

ZÜHLKE (P.). See LIETZMANN (W.).

III. APPLIED MATHEMATICS.

BLANCHARD (A. H.). American highway engineers' handbook. By A. H. Blanchard and seventeen associate editors. New York, Wiley, 1919. 25 + 1658 pp. $5.00

BRESTER (A.). A summary of my theory of the sun. The Hague, W. P. Stockum, 1919. 62 pp.

BUE (A. DEL). Lezioni di meccanica generale. Parte 2: Dinamica. Città di Castello, Soc. tip. Leonardo da Vinci, 1919. 8vo. 171 pp. L. 6.50

COMSTOCK (G. C.). The Sumner line or line of position as an aid to navigation. New York, Wiley, 1919. 6 + 70 pp. 12mo. $1.25

FRENCH (J. W.). See STEINHEIL (A.).

FULLER (C. E.) and JOHNSTON (W. A.). Applied mechanics. Volume 2: Strength of materials. New York, Wiley, 1919. 11 + 556 pp. $3.75

GARUFFA (E.). L'aviazione: aeroplani, idrovolanti, eliche. 2a edizione, interamente rifatta e aumentata. (Manuali Hoepli.) Milano, Hoepli, 1919. 24mo. 23 + 915 pp. L. 20.00

———. Turbine a vapore, con un capitolo sulle turbine a gas: teoria, calcolazione, costruzioni. (Manuali Hoepli.) Milano, Hoepli, 1919. 24mo. 15 + 782 pp. + tavola. L. 22.50

HARTMANN (O.). Astronomische Erdkunde. 5te Auflage. Stuttgart, Grub, 1918. 83 pp.

JOHNSTON (W. A.). See FULLER (C. E.).

LORIA (G.). Metodi de geometria descrittiva. 2a edizione, riveduta e migliorata. (Manuali Hoepli.) Milano, Hoepli, 1919. 24mo. 20 + 353 pp. L. 6.00

MACFARLANE (A.). Lectures on ten British physicists of the nineteenth century. (Mathematical Monographs, No. 20.) New York, Wiley, 1919. 143 pp. $1.50

PERNOT (F. E.). Electrical phenomena in parallel conductors. Volume 1: Elements of transmission. New York, Wiley, 1918. 12 + 332 pp. $4.00

SMITH (O. S.). Arithmetic of business. Chicago, Lyons and Carnahan, 1917. 8vo. 6 + 448 pp. $1.10

SOUTHALL (J. P. C.). Mirrors, prisms and lenses. New York, Macmillan, 1918.

STEINHEIL (A.) and VOIT (E.). Applied optics: the computation of optical systems; edited by J. W. French from the "Handbuch der angewandten Optik." Volume 2. London, Blackie, 1919. 213 pp. 12s. 6d.

VERHOECKX (P. M.). De vierdimensionale wereld der relativistische natuurkunde. Maestricht, typographie de Leiter-Nypels, 1918. 4to. 56 pp.

VOIT (E.). See STEINHEIL (A.).